Four Lectures on Polynomial Optimization for Control

Lecture 2: Sum-of-Squares Lyapunov methods

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Motivation

- Nonlinear control systems are generally **hard to analyse**
- Control synthesis is even harder
- Does the moment-SOS hierarchy help for polynomial systems?

- **Lyapunov function for system $\dot{x} = f(x)$:**

$$V(x) > 0 \quad \text{for } x \neq 0,$$

$$V(0) = 0,$$

$$\left(\frac{\partial V}{\partial x}\right)^\top f(x) < 0, \quad \text{for all } x \in \mathbb{R}^n \backslash \{0\}.$$

- **Central idea:** Replace the above with SOS conditions.

Polynomial dynamical system

$$
\begin{aligned}
\dot{x} &= f_x(x,u) \\
\text{with constraints} \quad a_{i_1}(x,u) &\le 0, \quad i_1 = 1,\dots,N_1, \\
b_{i_2}(x,u) &= 0, \quad i_2 = 1,\dots,N_2, \\
\int_0^T c_{i_3}(x,u)\,\mathrm{d}t &\le 0, \quad i_3 = 1,\dots,N_3,\ \forall T \ge 0,
\end{aligned}
$$

where

- $x \in \mathbb{R}^n$ is the state vector,
- $u \in \mathbb{R}^m$ is an *auxiliary vector* containing any of:
  - control inputs,
  - non-polynomial functions of states,
  - uncertain parameters,
- all functions a_{i_1}, b_{i_2}, c_{i_3} are polynomial in (x,u),
- the function $f_x(x,u)$ is either
  - a vector of polynomial functions,
  - a vector of ratio-of-polynomial functions $\frac{n(x,u)}{d(x,u)}$ with no singularity on

$$\mathcal{D} := \{(x,u) \in \mathbb{R}^{n+m} \mid a_{i_1}(x,u) \le 0,\ b_{i_2}(x,u) = 0,\ \forall i_1, i_2\}$$

Lyapunov analysis

Assume that (x, u) have been defined such that $f_x(0, u) = 0$ for $x = 0$ and $u \in \mathcal{D}_u^0 := \{u \,|\, (0, u) \in \mathcal{D}\}$. Then the following result holds.

Theorem (PP2002, Theorem 1)

Suppose that for the above system there exist polynomial functions $V(x)$, $w(x, u)$, $p_{i_1}(x, u)$, $q_{i_2}(x, u)$, and constants $r_{i_3} \geq 0$, such that $V(x)$ is positive definite, and such that $w(x, u) > 0$ and $p_{i_1}(x, u) \geq 0$ in $\mathcal{D}$.

If either of the following two conditions hold:

(i)

$$-\frac{\partial V}{\partial x} \cdot f_x(x, u) + \sum_{i_1=1}^{N_1} p_{i_1}(x, u) a_{i_1}(x, u) + \sum_{i_2=1}^{N_2} q_{i_2}(x, u) b_{i_2}(x, u) + \sum_{i_3=1}^{N_3} r_{i_3} c_{i_3}(x, u) \geq 0$$

(ii)

$$-w(x, u) \frac{\partial V}{\partial x} \cdot f_x(x, u) + \sum_{i_1=1}^{N_1} p_{i_1}(x, u) a_{i_1}(x, u) + \sum_{i_2=1}^{N_2} q_{i_2}(x, u) b_{i_2}(x, u) \geq 0$$

then $x = 0$ is a stable equilibrium of the system.

Lyapunov analysis

Proof.

Integrating both inequalities from $t = 0$ to $t = T$ yields

(i)

$$V(0) - V(T) \geq \int_0^T \left(\sum_{i_1=1}^{N_1} p_{i_1}(x,u) a_{i_1}(x,u) + \sum_{i_3=1}^{N_3} r_{i_3} c_{i_3}(x,u) \right) \mathrm{d}t \geq 0.$$

(ii)

$$w(x,u)(V(0) - V(T)) \geq -\sum_{i_1=1}^{N_1} \int_0^T p_{i_1}(x,u) a_{i_1}(x,u) \, \mathrm{d}t \geq 0.$$

which implies

$$V(0) - V(T) \geq -\sum_{i_1=1}^{N_1} \int_0^T \frac{p_{i_1}(x,u) a_{i_1}(x,u)}{w(x,u)} \, \mathrm{d}t \geq 0.$$

Thus, $x = 0$ is a stable equilibrium by the standard Lyapunov argument. □

Additional technicalities apply if $f_x(x,u)$ is a rational vector field [PP2002, Remark 2].

Finding polynomials $V(x)$, $w(x,u)$, $p_{i_1}(x,u)$, $q_{i_2}(x,u)$, and constants r_{i_3}

$$
\begin{aligned}
\dot{x} &= f_x(x,u) \\
\text{with constraints} \quad a_{i_1}(x,u) &\le 0, \quad i_1 = 1,\dots,N_1, \\
b_{i_2}(x,u) &= 0, \quad i_2 = 1,\dots,N_2, \\
\int_0^T c_{i_3}(x,u)\,\mathrm{d}t &\le 0, \quad i_3 = 1,\dots,N_3,\ \forall T \ge 0,
\end{aligned}
$$

The theorem above guarantees stability of the origin. But the certificate relies on choosing a number of polynomials:

- $V(x)$ where $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$;
- $w(x,u) > 0$ for all $(x,u) \in \mathcal{D}$;
- $p_{i_1}(x,u) \ge 0$ for all $(x,u) \in \mathcal{D}$ (associated with constraints $a_{i_1}(x,u) \le 0$);
- $q_{i_2}(x,u)$ of indefinite sign (associated with constraints $b_{i_2}(x,u) = 0$);
- $r_{i_3} \ge 0$ (associated with constraints $\int_0^T c_{i_1}(x,u)\,\mathrm{d}t \le 0$);

Sum-of-Squares Lyapunov function

These polynomials can be found via a SOS program:

SOS Lyapunov function

$$
\begin{aligned}
\min_{V, W, \{p_{i_1}\}, \{q_{i_2}\}, \{r_{i_3}\}} \quad & 0 \\
\text{s. t.} \quad & V - W \in \Sigma[x], \\
& W \in \mathcal{F}, \\
& \text{Left-hand side of condition (i) or (ii)} \in \Sigma[x], \\
& p_{i_1} \in \Sigma[x], \quad i_1 = 1, \ldots, N_1, \\
& r_{i_3} \geq 0, \quad i_3 = 1, \ldots, N_3.
\end{aligned}
$$

- Set $\mathcal{F}$ is a pre-chosen positive definite form for the polynomial $W(x)$.[1]
- The SOS program is solved for a particular level d of the SOS hierarchy.
- A solution for given d is **sufficient** to certify stability of the origin.
- If there is no solution, this does not imply the system is not stable.
- Can increase d to expand class of Lyapunov functions sought.

[1] For example $W(x) = \sum_{k=1}^{n} \varepsilon_k x_k^2$ with $\varepsilon_k \geq 0.1$ for all k.

Example: Polynomial dynamics in $\mathbb{R}^6$

$$\begin{aligned}
\dot{x}_1 &= -x_1^3 + 4x_2^3 - 6x_3x_4 \\
\dot{x}_2 &= -x_1 - x_2 + x_5^3 \\
\dot{x}_3 &= x_1x_4 - x_3 + x_4x_6 \\
\dot{x}_4 &= x_1x_3 + x_3x_6 - x_4^3 \\
\dot{x}_5 &= -2x_2^3 - x_5 + x_6 \\
\dot{x}_6 &= -3x_3x_4 - x_5^3 - x_6
\end{aligned}$$

This system has an equilibrium at $x = 0$, and no u, nor any constraints a_{i_1}, b_{i_2}, c_{i_3}.

Observations:

- The SOS program is simply "find V, W such that $V - W \in \Sigma[x]$, $W \in \mathcal{F}$, and $-\frac{\partial V}{\partial x} f_x(x) \in \Sigma[x]$."
- For level $d = 1$, i.e. quadratic V and W, with $\mathcal{F}$ chosen as in footnote 1, the SOS program has no solution.
- For level $d = 2$, i.e. quartic V and W, where W belongs to the class $\mathcal{F} = \{\sum_{k=1}^{6}(\varepsilon_{1k}x_k^2 + \varepsilon_{2k}x_k^4) \,|\, \varepsilon_{1k} + \varepsilon_{2k} \geq 0.1 \ \forall k\}$, a Lyapunov function is found,

$$V(x) = 0.7257x_1^2 + 1.3x_2^4 + 2.325x_3^2 + 1.575x_4^2 + 0.65x_5^4 + 1.3x_6^2$$

Auxiliary variables u

Auxiliary variables can be used to represent

- control inputs,
- non-polynomial functions of states,
- uncertain parameters,
- equilibrium locations.

Additional constraints coupling x and u can specify, for example, how an equilibrium depends on another parameter.

For example, a chemical reaction between two species with concentrations u and v:

$$\begin{aligned}\dot{u} &= a - u + u^2 v \\ \dot{v} &= b - u^2 v\end{aligned}$$

where $a > 0$ and $b > 0$ are unknown concentrations of two other species affecting u and v. Any equilibrium $(\bar{u}, \bar{v})$ must satisfy

$$0 = a - \bar{u} + \bar{u}^2 \bar{v} \tag{s1}$$

$$0 = b - \bar{u}^2 \bar{v} \tag{s2}$$

Then we can define $(u_1, u_2, u_3, u_4) := (a, b, \bar{u}, \bar{v})$, (s1) and (s2) become equality constraints of the form $b_1(x, u) = 0$ and $b_2(x, u) = 0$.

Auxiliary variables u

Furthermore, uncertainty for parameters $u_1 = a$ and $u_2 = b$ can be encoded with inequality constraints:

$$0 \geq \underline{a} - u_1$$
$$0 \geq \underline{b} - u_2$$

The state variables (x_1, x_2) are then redefined relative to the equilibrium $(u_3, u_4) = (\bar{u}, \bar{v})$:

$$\dot{x}_1 = u_1 - (x_1 + u_3) + (x_1 + u_3)^2(x_2 + u_4)$$
$$\dot{x}_2 = u_2 - (x_1 + u_3)^2(x_2 + u_4)$$

Auxiliary variables u

Final system with constraints, where:

- $(x_1, x_2) = (u - \overline{u}, v - \overline{v})$ and
- $(u_1, u_2, u_3, u_4) = (a, b, \overline{u}, \overline{v})$

$$
\begin{aligned}
\dot{x}_1 &= u_1 - (x_1 + u_3) + (x_1 + u_3)^2 (x_2 + u_4) \\
\dot{x}_2 &= u_2 - (x_1 + u_3)^2 (x_2 + u_4) \\
0 &\geq \underline{a} - u_1 \qquad =: a_1(x, u) \\
0 &\geq \underline{b} - u_2 \qquad =: a_2(x, u) \\
0 &= u_1 - u_3 + u_3^2 u_4 \qquad =: b_1(x, u) \\
0 &= u_2 - u_3^2 u_4 \qquad =: b_2(x, u)
\end{aligned}
$$

In addition, restrict search to a local equilibrium: $|u - \overline{u}| < \gamma \overline{u}$ and $|v - \overline{v}| < \gamma \overline{v}$ for $0 < \gamma \leq 1$. This leads to two more constraints $a_3(x, u) = x_1^2 - \gamma u_3^2 \leq 0$ and $a_4(x, u) = x_2^2 - \gamma u_4^2 \leq 0$.

$\Rightarrow$ can then solve SOS program to certify stability for entire ranges of uncertain a and b.

Non-polynomial dynamics

The auxiliary variables u can also be used to model non-polynomial dynamics, with additional constraints. For example the “whirling pendulum” of [PP2002, Fig. 4]:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \dot{\theta}_a^2 \sin x_1 \cos x_2 - \frac{g}{l_p} \sin x_1.\end{aligned}$$

With the substitution $u_1 = \sin x_1$, $u_2 = \cos x_1$ one obtains

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \dot{\theta}_a^2 u_1 u_2 - \frac{g}{l_p} u_1.\end{aligned}$$

with the additional consistency constraint $u_1^2 + u_2^2 = 1$. This is called a “lifting” to higher dimension, as more variables are introduced than are needed to model the system.

The advantage of the new polynomial representation is that the SOS framework can then be used to find a Lyapunov function for the lifted problem.

Control synthesis

So far, we have only looked at autonomous systems, essentially $\dot{x} = f_x(x)$. How do we use this framework to design a controller for

- Stability?
- Closed-loop performance?

Scope

Nonlinear systems of the "state-dependent linear" form

$$\dot{x} = A(x)\mathbf{x} + B(x)u,$$

where $A(x)$ and $B(x)$ are polynomial matrices in x. We seek a controller

$$u = F(x)\mathbf{x},$$

where $\mathbf{x}$ is a vector of monomials in x satisfying

$$\mathbf{x} = 0 \Leftrightarrow x = 0.$$

Lyapunov theorem for polynomial control synthesis

Let $\tilde{x}$ be the sub-vector of x corresponding to the rows of $B(x)$ whose entries are all zero, and let $\mathcal{J}$ be the set of indices for the zero rows in $B(x)$.

Theorem (PPW2004, Theorem 6)

Suppose there exist appropriately-dimensioned polynomial matrix $K(x)$ and symmetric polynomial matrix $P(\tilde{x})$, a constant $\varepsilon_1 > 0$, and a $\varepsilon_2(x) \in \Sigma[x]$, such that

$$v^\top (P(\tilde{x}) - \varepsilon_1 I) v \in \Sigma[x, v]$$

$$\text{and} \quad -v^\top \Big(P(\tilde{x}) A^\top(x) M^\top(x) + M(x) A(x) P(\tilde{x}) + K^\top(x) B^\top(x) M^\top(x) + M(x) B(x) K(x) - \sum_{j \in \mathcal{J}} \frac{\partial P}{\partial x_j}(\tilde{x}) A_j(x) x + \varepsilon_2(x) I \Big) v \in \Sigma[x, v],$$

where $\Sigma[x, v]$ is the set of polynomials of the form $v^\top F(x) v$ that can be written $(v \otimes \mathbf{x})^\top Q (v \otimes \mathbf{x})$, for some symmetric $Q \succeq 0$. Recall that $\mathbf{x}$ denotes the vector of monomials in x. Matrix $M(x)$ is defined by $M_{ij}(x) = \partial \mathbf{x}_i / \partial x_j$.

Then $u(x) = K(x) P^{-1}(\tilde{x}) \mathbf{x}$ is **locally** *stabilizing.*

If $\varepsilon_2(x) > 0$ for $x \neq 0$ then the controller is **asymptotically stabilizing**.

Kronecker product SOS condition

The theorem uses a sum-of-squares restriction of constraints of the form

$$v^\top F(x)v \geq 0, \quad \forall v \in \mathbb{R}^n,$$

where $F(x)$ is *polynomial* in x. The usual condition $F(x) \succeq 0 \ \forall x \in \mathbb{R}^n$ is not computationally efficient to work with. However there is a sufficient SOS representation

$$v^\top F(x)v = (v \otimes \mathbf{x})^\top Q(v \otimes \mathbf{x}),$$

for some symmetric $Q \succeq 0$. The symbol $\otimes$ denotes the Kronecker product. For $v \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^{s(d)}$, this is given by

$$v \otimes x := \begin{bmatrix} v_1 \cdot 1 \\ v_1 x_1 \\ \vdots \\ v_1 x_n^d \\ \vdots \\ v_n \cdot 1 \\ v_n x_1 \\ \vdots \\ v_n x_n^d \end{bmatrix}$$

Lyapunov theorem for polynomial control synthesis

A more restrictive formulation requires a constant P matrix, but guarantees global stability.

Theorem (PPW2004, Theorem 6: Global stability version)

Suppose there exist $n \times n$ *symmetric* **constant** *matrix* P *and polynomial* $K(x)$, *a constant* $\varepsilon_1 > 0$, *and a* $\varepsilon_2(x) \in \Sigma[x]$ *such that*

$$v^\top (P - \varepsilon_1 I) v \in \Sigma[x, v]$$

and

$$-v^\top \Big(P A^\top(x) M^\top(x) + M(x) A(x) P + K^\top(x) B^\top(x) M^\top(x) + M(x) B(x) K(x) + \varepsilon_2(x) I \Big) v \in \Sigma[x, v].$$

Then $u(x) = K(x) P^{-1} x$ *is* **globally** *stabilizing.*

If $\varepsilon_2(x) > 0$ *for* $x \neq 0$ *then the controller is* **asymptotically stabilizing**.

$\mathcal{H}_\infty$ control synthesis

SOS techniques can be used to design a controller for the polynomial system

$$\begin{aligned}
\dot{x} &= A(x)\mathbf{x} + B_1(x)w + B_2(x)u \\
z_1 &= C_1(x)\mathbf{x} \\
z_2 &= C_2(x)\mathbf{x} + u
\end{aligned}$$

where $z_1 \in \mathbb{R}^{M_1}$ are outputs and $w \in \mathbb{R}^{M_2}$ are disturbances.

We are interested in minimizing the largest induced $\mathcal{L}_2$ gain from w to z, (cf. definition of $\mathcal{H}_\infty$ norm in linear control design: $\|G(s)\|_\infty = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}$).

We will synthesize a controller of the form

$$u(x) = -[\gamma B_2^\top(x) P^{-1}(\tilde{x}) + C_2(x)]\mathbf{x},$$

where $\tilde{x}$ and the associated index set $\mathcal{J}$ now correspond to the zero rows of $[B_1(x)\; B_2(x)]$, and γ is the lowest upper bound on $\mathcal{L}_2$ gain achieved.

$\mathcal{H}_\infty$ control synthesis for polynomial systems

Theorem (PPW2004, Theorem 9)

Suppose there exist a symmetric polynomial matrix $P(\tilde{x})$*, a constant* $\varepsilon_1 > 0$*, and a* $\varepsilon_2(x) \in \Sigma[x]$*, such that*

$$v^\top (P(\tilde{x}) - \varepsilon_1 I) v \in \Sigma[x, v]$$

and

$$-\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^\top \left[\begin{array}{c|c|c} \begin{matrix} M\hat{A}P + P\hat{A}^\top M^\top - \gamma M B_2 B_2^\top M^\top \\ -\sum_{j \in \mathcal{J}} \frac{\partial P}{\partial x_j} A_j + \varepsilon_2 I \end{matrix} & PC_1^\top & MB_1 \\ \hline C_1 P & -(\gamma - \varepsilon_2) I & 0 \\ \hline B_1^\top M^\top & 0 & -(\gamma - \varepsilon_2) I \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \Sigma[x, v],$$

where some matrices' dependence on x *is omitted for brevity, and* $\hat{A}(x) := A(x) - B_2(x) C_2(x)$.

Then $u(x) = -[\gamma B_2^\top(x) P^{-1}(\tilde{x}) + C_2(x)]\mathbf{x}$ *has* $\mathcal{L}_2$ *gain* **locally** *bounded by* γ*.*
If $P(\tilde{x})$ *is a constant matrix, then the bound of* γ *holds* **globally** *for* $x \in \mathbb{R}^n$*.*

Synthesis problem is therefore "minimize γ subject to constraints in Theorem 9".

$\mathcal{H}_\infty$ control synthesis for polynomial systems

In addition, can design a *local* controller operating within some region

$$\mathcal{X} := \{x \in \mathbb{R}^n \mid g_l(x) \geq 0,\ l = 1, \ldots, s\}$$

by inserting additional SOS mulipliers for the constraints $g_l(x) \geq 0$.

Theorem (PPW2004, Proposition 11)

Suppose the SOS conditions of Theorem 6 are augmented

$$v^\top (P(\tilde{x}) - \varepsilon_1 I)v - \sum_{l=1}^{s} \sigma_{1,l}(x,v) g_l(x) \in \Sigma[x,v]$$

$$-v^\top \Big(P(\tilde{x}) A^\top(x) M^\top(x) + \cdots + \varepsilon_2(x) I \Big) v - \sum_{l=1}^{s} \sigma_{2,l}(x,v) g_l(x) \in \Sigma[x,v],$$

where the degree of $\sigma_{1,l}$ and $\sigma_{2,l}$ in v is equal to two.

Then $u(x) = K(x)P^{-1}(\tilde{x})x$ is stabilizing, and an estimate of the domain of attraction of the closed-loop zero equilibrium is

$$\{x \mid x^\top P^{-1}(\tilde{x})x \leq \overline{V}\},$$

where $\overline{V}$ is the smallest value of $x^\top P^{-1}(\tilde{x})x$ achieved on the boundary of $\mathcal{X}$.

Notes and references

Toolbox

SOSTOOLS: `http://www.cds.caltech.edu/sostools/`

Literature

This lecture was based on the following well-known papers:

A. Papachristodoulou and S. Prajna, "On the Construction of Lyapunov Functions using the Sum of Squares Decomposition," IEEE Conf. on Decision and Control, Las Vegas, NV, USA, 2002

S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear Control Synthesis by Sum of Squares Optimization: A Lyapunov-based Approach," Asian Control Conference, 2004

Early formulations are to be found in the PhD thesis
P. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Caltech, 2000.